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GAUGE BOSON AND FERMION MASSES WITHOUT A HIGGS FIELD

Angus F. Nicholson¹

*Department of Theoretical Physics, IAS
The Australian National University, Canberra, ACT 0200, Australia*

Dallas C. Kennedy²

*Department of Physics, University of Florida,
Gainesville, Florida 32611, USA*

ABSTRACT

A simple, anomaly-free chiral gauge theory can be perturbatively quantised and renormalised in such a way as to generate fermion and gauge boson masses. This development exploits certain freedoms inherent in choosing the unperturbed Lagrangian and in the renormalisation procedure. Apart from its intrinsic interest, such a mechanism might be employed in electroweak gauge theory to generate fermion and gauge boson masses without a Higgs sector.

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¹Deceased

²e-mail: kennedy@phys.ufl.edu

1 Introduction

There is a widespread, plausible view that the Standard Model is a low-energy effective theory and that the Higgs boson might not exist [1]. A Higgs sector introduces naturalness, triviality and cosmological problems [2]. The simple model given in this paper illustrates a perturbative method for giving mass to gauge bosons and fermions with chiral couplings which does not require Higgs fields or new fermions.

The mass mechanism presented here might be used in a fundamental theory or in a low-energy effective field theory. An effective theory need not be renormalisable [3]. Our model gives mass to the gauge boson on renormalisation to one loop, is renormalisable to several loops or orders, and appears to possess QED-like renormalisability to all orders. The model branches into three cases, in which the masses are $O(g^0)$, $O(g^1)$ (like m_W , m_Z in standard electroweak theory) and $O(g^2)$. The mechanism seems to be applicable to any chiral gauge theory.

The effective Lagrangian density of the model, including gauge-fixing and ghost terms, is

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{4}Z_3(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - (2\xi)^{-1}Z_4(\partial_\mu A^\mu)^2 + (\partial_\mu c^*)\partial^\mu c \\ & + Z_{2L}\hbar\bar{\psi}_L i\not{\partial}\psi_L + Z_{2R}\hbar\bar{\psi}_R i\not{\partial}\psi_R - \hat{g}\bar{\psi}\gamma^\mu[Z_1 + fZ_{1(5)}\gamma^5]\psi A_\mu \\ & + Z_{6L}\hbar\bar{\eta}_L i\not{\partial}\eta_L + Z_{6R}\hbar\bar{\eta}_R i\not{\partial}\eta_R + \hat{g}\bar{\eta}\gamma^\mu[Z_5 + fZ_{5(5)}\gamma^5]\eta A_\mu, \end{aligned} \quad (1)$$

which has been obtained from a chiral- and gauge-invariant classical \mathcal{L} in the path integral formalism. A renormalisation Z_4 of the gauge-fixing term is allowed. We have included \hbar , dimensionless but not yet numerically 1, since \hbar^{-1} is the expansion parameter in our Case B (see section 2).

There are no mass terms in $A_\mu A^\mu$, $\bar{\psi}\psi$ or $\bar{\eta}\eta$ in \mathcal{L}_{eff} , and in section 2 we see that the effective action $S_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}}$ is invariant under $U(1)_{L,R}$ BRS transformations. For $f \neq 0$, its $U(1)_{L,R}$ invariance does not permit \mathcal{L} or \mathcal{L}_{eff} any mass terms. It might then appear that \mathcal{L}_{eff} must define a massless theory. However, we recall nonperturbative counterexamples. The standard electroweak Lagrangian in its initial form does not contain vector boson mass terms of the form $M^2 A_\mu A^\mu$, yet the theory gives massive W , Z bosons by spontaneous symmetry breaking (SSB). For the QED Lagrangian

$$\tilde{\mathcal{L}} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\not{\partial} - e\not{A})\psi \quad (2)$$

in $1+1$ spacetime dimensions, the Schwinger model [4] has an exact solution in which the only physical particle is a massive vector boson [5,6], the

mass arising from quantum corrections. Other nonperturbative examples with initially massless fermions and dynamical breaking of chiral symmetry are known. Nambu and Jona-Lasinio (NJL) [7] described nucleons with a four-fermion interaction, a mechanism natural with massive gauge boson exchange. Baker *et al.* and Green *et al.* [8] obtained from massless QED asymptotic and full forms, respectively, of a propagator approximating the standard form for a massive fermion. Jackiw and Johnson [9] and Cornwall and Norton [10] generated fermion and gauge masses by massless bound $f\bar{f}$ excitations. More recently, Roberts and others [11,12,13] have obtained quark self-energies from truncated Dyson–Schwinger equations for QCD, breaking the chiral symmetry dynamically. Gusynin, Miransky and others [14,15] have applied nonperturbative methods to the NJL model and QED and obtained dynamical fermion masses for an arbitrarily weak attraction between fermions. In contrast to these nonperturbative developments, we show that \mathcal{L}_{eff} can give fermions and gauge bosons masses *perturbatively*. We make an unconventional ansatz for \mathcal{L}_0 in the decomposition $\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_1$ and renormalise in a novel way.

Gauge symmetry is replaced by BRS symmetry because \mathcal{L}_{eff} contains a gauge-fixing term and ghosts. Then BRS symmetry is broken upon making the decomposition $\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_1$. While the effective action $S_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}}$ is BRS-invariant in QED, QCD and this model, the partial-actions $S_{0,1} = \int d^4x \mathcal{L}_{0,1}$ are not. This lack of gauge symmetry in \mathcal{L}_0 does not matter, provided that the choice of \mathcal{L}_0 gives S -matrix elements independent of the gauge (an issue not fully resolved [16]). In this model \mathcal{L}_0 also does not possess the chiral invariance of \mathcal{L}_{eff} . However, the propagators and vertices that we obtain, and so the S -matrix elements, do not depend on the chirality phase, in the way that propagators in QED and QCD usually depend on a gauge parameter ξ .

It is conventional to take as \mathcal{L}_0 the quadratic part $\mathcal{L}_{\text{quad}}$ of a given \mathcal{L} , rather than to quantise from another \mathcal{L}_0 , i.e., from $\mathcal{L}_0 = \mathcal{L}_{\text{quad}} + \chi$. For example, given the form (2), it would be normal to take $\tilde{\mathcal{L}}_0$ to be the quadratic part of $\tilde{\mathcal{L}}$ and expect the result to be massless QED. Yet the examples cited above show that a theory massless in its quadratic part can nonetheless generate masses dynamically once interactions are introduced. We take the view that any choice of \mathcal{L}_0 that leads to a self-consistent model is legitimate, especially if the mass mechanism proposed gives promise of being applicable in a physical theory, such as an electroweak theory without a Higgs sector.

We use a decomposition of \mathcal{L} in which \mathcal{L}_0 contains fermion and gauge

boson mass terms. The mass term

$$\chi = \frac{1}{2}\hbar^{-2}m_A^2 A_\mu A^\mu - m_1 \bar{\psi}\psi - m_2 \bar{\eta}\eta, \quad (3)$$

is placed in \mathcal{L}_0 and $-\chi$ in \mathcal{L}_1 , so that \mathcal{L}_{eff} is unchanged and the $U(1)_{L,R}$ invariance is unaffected. Splitting a zero term in \mathcal{L}_{eff} to obtain $\chi, -\chi$ in $\mathcal{L}_0, \mathcal{L}_1$ is akin to splitting $m_0 \bar{\psi}\psi$ into $m_{\text{phys}} \bar{\psi}\psi$ and $\delta m \bar{\psi}\psi$ in QED. We make the choice

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - (2\xi)^{-1}(\partial_\mu A^\mu)^2 + \frac{1}{2}\hbar^{-2}m_A^2 A_\mu A^\mu + (\partial_\mu c^*)\partial^\mu c \\ & + \bar{\psi}(i\hbar\partial - m_1)\psi + \bar{\eta}(i\hbar\partial - m_2)\eta \end{aligned} \quad (4)$$

(since $\psi_L i\partial\psi_L + \bar{\psi}_R i\partial\psi_R = \bar{\psi}i\partial\psi$, and similarly for η), which generates massive particles, and show that we can regenerate dynamically the masses m_1, m_2 , and m_A or generate a complex mass for the boson (when it is unstable), to obtain a self-consistent theory. We begin from \mathcal{L}_{eff} , which by itself does not necessarily exhibit the physical spectrum.

It might appear that the $(\chi, -\chi)$ step must be nugatory, since the boson and fermion mass terms from χ in \mathcal{L}_0 and $-\chi$ in \mathcal{L}_1 cancel in the denominators of the full, improper propagators, e.g., in $\{\not{p} - m - [-m + \Sigma(\not{p}) + c.t.]\}$. However, m contributes to $\Sigma(\not{p})$, and this leads to nonzero renormalised fermion masses. The m_1, m_2 terms in the fermion propagators that follow from (4) (in section 2) are essential for the generation of a $g_{\mu\nu}\rho(k^2)$ term in the boson self-energy tensor $\pi_{\mu\nu}(k)$, which is

$$\pi_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)\pi(k^2) + k_\mu k_\nu \tau(k^2) + g_{\mu\nu}\rho(k^2), \quad (5)$$

where, to avoid ambiguity, $\rho(k^2)$ is defined not to contain a factor k^2 . It is this term that leads to the mass of the gauge boson in our model. Similar mass terms appear in $\pi_{\mu\nu W}, \pi_{\mu\nu Z}$ in standard electroweak theory [17]. The nonzero fermion and boson mass ansatz can be maintained self-consistently: the loops regenerate the masses, or generate a complex boson mass, so that they are quantum-field-theoretical in origin.

So that the series for the full boson and fermion propagators can be summed in the usual way, correctly to any given power of the expansion parameter, each term in \mathcal{L}_1 , and so m_1, m_2, m_A in $-\chi$, must be proportional to a positive power of that parameter. In Case A1 and Case A2 the expansion parameter is g . In Case A1 we take

$$\text{A1 : } m_1 = \beta_1 g, \quad m_2 = \beta_2 g, \quad m_A = \beta_A g, \quad (6)$$

and on renormalisation obtain a boson mass of $O(g)$. We recall that m_W , m_Z are $O(g)$ in standard electroweak theory, and that the boson mass is $e\pi^{-1/2}$ in the Schwinger model. In Case A2 we take

$$\text{A2: } m_1 = \beta_1 g^2, \quad m_2 = \beta_2 g^2, \quad m_A = \beta_A g^2, \quad (7)$$

and obtain a boson mass of $O(g^2)$. In both these cases we could put $\hbar = 1$ at the beginning. In Case B the expansion parameter is $w = \hbar^{-1}$ (see section 2) and we put $\hbar = 1$ at the end of the renormalisation. We take

$$\text{B: } m_1 = \beta_1 w, \quad m_2 = \beta_2 w, \quad m_A = \beta_A w, \quad (8)$$

and the final boson mass is $O(w) = O(wg^0)$, i.e., $O(1)$ when we put $\hbar = 1$. Thus our three cases cover masses of $O(1)$, $O(g)$ and $O(g^2)$.

We employ dimensional regularisation, with γ^5 totally anticommuting in $d = 4 - 2\epsilon$ dimensions [18,19,20], a standard regularisation method [17,21,22,23,24]. At one loop, the divergent part of $\rho(k^2)$ is

$$\rho_{2\epsilon} = \frac{\hbar^{-2} g^2 f^2}{2\pi^2 \epsilon} (m_1^2 + m_2^2). \quad (9)$$

Since no gauge boson mass counterterm can appear in \mathcal{L}_{eff} , $\rho_{2\epsilon}$ cannot be cancelled. Further, $\rho_{2\epsilon}$ vanishes as $f \rightarrow 0$: the generation of gauge boson mass is chiral in this model.

If the renormalisation gives the boson a mass m_{AR} such that the boson is stable, $m_{AR} < 2m_1$, then the particles are stable, since the interactions in \mathcal{L}_1 ensure that the fermions are stable. Then unitarity requires that the LSZ S-matrix reduction formula [25] hold in the renormalised theory, so that the renormalised masses equal the masses in \mathcal{L}_0 , i.e., $m_{AR} = m_A$, $m_{1R} = m_1$, $m_{2R} = m_2$.

However, the only known massive elementary bosons are the W and the Z , so it is the unstable boson case that is of physical interest. Then the boson mass m_{AR} must be complex (e.g., for the Z boson [26,27]). But the usual perturbative or LSZ [28] formalisms do not accommodate unstable particles in a consistent way [29,30]. Questions of gauge invariance, unitarity, renormalisation and the complex Z mass pole are discussed in [31,32,33,34]. In this paper we use the standard perturbative formalism when there is an unstable particle present, but if our mass mechanism were used in a fundamental theory containing unstable particles, this formalism could not treat unitarity consistently. Moreover, predictions from an effective Lagrangian must be checked for unitarity at high energies [2]. We postpone further consideration of unitarity to a later publication.

2 $U(1)$ and Chiral Invariance, Propagators

Since we work in $d = 4 - 2\epsilon$ dimensions, \hat{g} in (1) is $\hat{g} = g\mu^\epsilon$, where g is dimensionless and μ is a scale mass; however, hereafter we drop this distinction, and \hat{g} is to be understood where appropriate. In the factors $Z_i = 1 - c_i$ in (1), the c_i are counterterm parameters. The primary interaction term is

$$\begin{aligned}\mathcal{L}_{1P} &= -g\bar{\psi}\gamma^\mu(1+f\gamma^5)\psi A_\mu + g\bar{\eta}\gamma^\mu(1+f\gamma^5)\eta A_\mu \\ &= -g[(1-f)\bar{\psi}_L\gamma^\mu\psi_L + (1+f)\bar{\psi}_R\gamma^\mu\psi_R]A_\mu \\ &\quad + g[(1-f)\bar{\eta}_L\gamma^\mu\eta_L + (1+f)\bar{\eta}_R\gamma^\mu\eta_R]A_\mu.\end{aligned}\tag{10}$$

The two anomalous fermion triangle divergences cancel, because of the $-g, g$ couplings to A_μ , and that the fermion quadrangle loops (applying an argument similar to that of [35] for QED) and higher polygon loops are convergent. In the limit of $f \rightarrow 0$, the theory becomes two-fermion QED with zero gauge boson mass.

It is straightforward to transform in the usual way to $\xi^0 = Z_4^{-1}Z_3\xi$, $A_\mu^0 = Z_3^{1/2}A_\mu$, $\psi_{L,R}^0 = Z_{2L,R}^{1/2}\psi_{L,R}$, $\eta_{L,R}^0 = Z_{6L,R}^{1/2}\eta_{L,R}$ and show that the action $S = \int d^4x \mathcal{L}_{\text{eff}}$ is invariant under the $U(1)_{L,R}$ BRS transformation

$$\left. \begin{aligned}A_\mu^0 &\rightarrow A_\mu^0 + \tau\partial_\mu c \\ c^* &\rightarrow c^* - \frac{1}{\xi^0}\tau\partial^\mu A_\mu^0, \quad c \rightarrow c \\ \psi_L^0 &\rightarrow (1 - i\hbar^{-1}g_{1L}^0\tau c)\psi_L^0, \quad \eta_L^0 \rightarrow (1 + i\hbar^{-1}g_{5L}^0\tau c)\eta_L^0 \\ \psi_R^0 &\rightarrow (1 - i\hbar^{-1}g_{1R}^0\tau c)\psi_R^0, \quad \eta_R^0 \rightarrow (1 + i\hbar^{-1}g_{5R}^0\tau c)\eta_R^0\end{aligned}\right\}\tag{11}$$

where τ is a Grassmann number and

$$\left. \begin{aligned}g_{1L}^0 &= (Z_1 - fZ_{1(5)})Z_{2L}^{-1}Z_3^{-1/2}g \\ g_{1R}^0 &= (Z_1 + fZ_{1(5)})Z_{2R}^{-1}Z_3^{-1/2}g \\ g_{5L}^0 &= (Z_5 - fZ_{5(5)})Z_{6L}^{-1}Z_3^{-1/2}g \\ g_{5R}^0 &= (Z_5 + fZ_{5(5)})Z_{6R}^{-1}Z_3^{-1/2}g.\end{aligned}\right\}\tag{12}$$

Accordingly, we have a $U(1)_{L,R}$ gauge theory. In addition, \mathcal{L} is invariant under the chirality transformation $\psi \rightarrow e^{i\alpha\gamma^5}\psi$, $\eta \rightarrow e^{i\beta\gamma^5}\eta$, with α, β constants. The $U(1)_{L,R}$ invariance, with $f \neq 0$, forbids mass terms. Since c^*, c do not couple to A_μ , we omit them from this point onwards. The BRS and chiral symmetries imply the usual Ward identities for (1).

Placing the mass term χ given by (3) in \mathcal{L}_0 and $-\chi$ in \mathcal{L}_1 does not affect the invariances of \mathcal{L} and the action. \mathcal{L}_0 does not possess the chiral invariance of \mathcal{L} , and, similarly to the situation in QED and QCD, the partial action $S_0 = \int d^4x \mathcal{L}_0$ does not possess the $U(1)_{L,R}$ invariance of S .

If the renormalisation is performed so as to give a stable boson of mass m_{AR} , we must have $m_{AR} = m_A$ and the renormalised fermion masses must be $m_{1R} = m_1$, $m_{2R} = m_2$. If the boson is to be unstable, the case of physical interest, then it must have a complex mass m_{AR} , which cannot equal a real m_A . Quantisation with any value for m_A gives a set of basis states which include a non-degenerate vacuum that possesses the symmetries of \mathcal{L}_0 .

In treatments of QED in which \hbar and c are dimensionful [36], \mathcal{L}_0 for a fermion is $\hbar c \bar{\psi} i \not{\partial} \psi - mc^2 \bar{\psi} \psi$, and diagrams are generated by the action of $(\hbar c)^{-1} \mathcal{L}_1$. Taking $c = 1$ and \hbar dimensionless but not yet 1 gives the forms of the fermionic parts in \mathcal{L}_{eff} , (1), and \mathcal{L}_0 , (4), and at each vertex we have $g\hbar^{-1}$. We obtain from (4), by the path integral method or canonical quantisation, the usual Fourier-transformed propagators

$$iD_{\mu\nu}(k) = \frac{-i\hbar}{k^2 - \mu^2 + i\epsilon'} \left[g_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2 - \xi\mu^2 + i\epsilon'} \right], \quad (13)$$

$$iS_{F\psi,\eta}(p) = i[\not{p} - \kappa_{1,2} + i\epsilon']^{-1}, \quad (14)$$

where $\mu = m_A \hbar^{-1}$, $\kappa_1 = m_1 \hbar^{-1}$ and $\kappa_2 = m_2 \hbar^{-1}$. In scalar φ^4 theory the propagator contains an \hbar factor (like that in $iD_{\mu\nu}$) which (with the \hbar^{-1} at each vertex included) leads to the “loop expansion” in powers of \hbar [37,38]. In this model, however, the absence of an \hbar factor in iS_F together with an \hbar^{-1} at each vertex causes loop integrals to contain factors $(\hbar^{-1})^n$, and it turns out that in Case B we must use $w = \hbar^{-1}$ as the expansion parameter. (We are not concerned with the classical limit $\hbar \rightarrow 0$, since we put $\hbar = 1$ on completion of the renormalisation in Case B and can put $\hbar = 1$ in Case A1 or A2 immediately.)

We refer to the $(k^2 - \mu^2)$ denominator and the pole at $k^2 = \mu^2$ in (13) as the primary denominator and primary pole of $D_{\mu\nu}$, and refer to the analogous objects in the improper and renormalised propagators $i\mathcal{D}_{\mu\nu}$, $i\mathcal{D}_{\mu\nu}^R$ by the same terms.

The topologies of the Feynman diagrams generated by $\hbar^{-1} \mathcal{L}_1$ are essentially the same as those of QED with two fermions, and, with the propagators (13), (14), the usual power-counting analysis [3,28] shows that the degree of divergence of an arbitrary diagram is the same as that of the analogous diagram in QED. The model is renormalisable. However, our procedure differs from the usual complete cancellation of divergences order by order [39].

3 Renormalisation of the Boson Propagator

The full boson propagator is

$$i\mathcal{D}_{\mu\nu} = iD_{\mu\nu} + iD_{\mu\sigma}[i\bar{\pi}^{\sigma\rho}]iD_{\rho\nu} + \dots \quad (15)$$

where $\bar{\pi}^{\sigma\rho}$, generated by $\hbar^{-1}\mathcal{L}_1$, is defined by

$$\bar{\pi}_{\mu\nu}(k) = \pi_{\mu\nu}(k) - c_3 w(k_\mu k_\nu - g_{\mu\nu} k^2) + c_4 w \xi^{-1} k_\mu k_\nu - m_A^2 w^3 g_{\mu\nu}, \quad (16)$$

where $w = \hbar^{-1}$ and $\pi_{\mu\nu}(k)$ is the self-energy tensor (5). For the lowest order ψ -loop and η -loop components we find, using (14), that $\rho(k^2) \neq 0$ if $f \neq 0$ (the divergent part is given by (9)), and that $\tau(k^2) = 0$ (as in QED).

In Case A1 or A2, each of m_A (by (6), (7)) and, it turns out below, $\pi_{\mu\nu}$, c_3 and c_4 are $O(g^j)$, $j \geq 1$, and we can sum the series (15) correctly to any given $O(g^n)$, in the usual way. In Case B the expansion parameter is w . It turns out that each term in (16) is of $O(w^j)$, $j \geq 1$, so that we can sum (15) correctly to any given $O(w^n)$. In every case we obtain

$$i\mathcal{D}_{\mu\nu}(k) = \frac{-iw^{-1}}{[1 + w^{-1}\pi(k^2) - c_3]k^2 - w^2 m_A^2 - w^{-1}[\rho(k^2) - w^3 m_A^2]} [g_{\mu\nu} + k_\mu k_\nu Q(k^2)], \quad (17)$$

where

$$Q(k^2) = \frac{\xi[1 + w^{-1}\pi(k^2) - c_3 + w^{-1}\tau(k^2)] + c_4 - 1}{k^2[1 - \xi w^{-1}\tau(k^2) - c_4] - \xi w^{-1}\rho(k^2)}, \quad (18)$$

correctly to $O(g^n)$ in Case A1 or A2, or $O(w^n)$ in Case B, for any given n . We see the expected cancellation in the primary denominator of $\mathcal{D}_{\mu\nu}$ between the $-m_A^2$, m_A^2 terms coming from \mathcal{L}_0 via (13), and \mathcal{L}_1 . In equations holding in all cases, or in A1 or A2, we usually write \hbar rather than w^{-1} in what follows, while in Case B we usually write \hbar as w^{-1} . The primary denominator of $\mathcal{D}_{\mu\nu}$ is, from (17),

$$d(k^2) = [1 + \hbar\pi(k^2) - c_3]k^2 - \hbar\rho(k^2). \quad (19)$$

We renormalise so as to place the zero of $d(k^2)$ at $k^2 = \mathcal{M}^2 = m_{AR}^2 \hbar^{-2}$, where m_{AR} is the renormalised boson mass, which may be complex if the boson is unstable. We write

$$\mathcal{M}^2 = M^2(1 - i\delta) \quad (20)$$

and expand $d(k^2)$ about $k^2 = \mathcal{M}^2$, to obtain

$$d(k^2) = e + \Omega_3^{-1}(k^2 - \mathcal{M}^2) + R(k^2), \quad (21)$$

where

$$e = [1 + \hbar \hat{\pi} - c_3] \mathcal{M}^2 - \hbar \hat{\rho}, \quad (22)$$

$$\Omega_3^{-1} = 1 + \hbar \hat{\pi} - c_3 + \mathcal{M}^2 \hbar \hat{\pi}' - \hbar \hat{\rho}' \quad (23)$$

and

$$\begin{aligned} R(k^2) &= \frac{\hbar}{2} (2\hat{\pi}' + \mathcal{M}^2 \hat{\pi}'' - \hat{\rho}'') (k^2 - \mathcal{M}^2)^2 \\ &\quad + \frac{\hbar}{6} (3\hat{\pi}'' + \mathcal{M}^2 \hat{\pi}''' - \hat{\rho}''') (k^2 - \mathcal{M}^2)^3 + \dots, \end{aligned} \quad (24)$$

where $\hat{\pi} = \pi(\mathcal{M}^2)$, $\hat{\rho} = \rho(\mathcal{M}^2)$, $\hat{\pi}', \hat{\rho}', \dots$ are defined by (A.12), (A.13) in Appendix A. Ω_3 is the renormalisation factor multiplying the renormalised propagator $i\mathcal{D}_{\mu\nu}^R(k)$ (the analogous factor in QED is commonly written as Z_3).

From (5) we see that $k^2 \pi(k^2)$, $\rho(k^2)$ are of the same dimension, and with $c = 1$ and \hbar dimensionless, k^2 has the dimension of m^2 , so that we can write

$$\rho(k^2) = \sum m_i m_j \sigma^{ij}(k^2) \quad (25)$$

with $\sigma^{ij}(k^2)$, $\pi(k^2)$ of the same dimension and with their components having the same w and g dependence (see Appendix A). m_i, m_j stand for m_1, m_2, m_A in all combinations; at least the m_1^2, m_2^2 terms are not zero.

3.1 Real mass, $\mathcal{M} = M$

To obtain a renormalised real mass $m_{AR} = \hbar M$ we proceed loop by loop, in the n -loop and $(n+)$ -loop sets of diagrams defined and discussed in Appendix A. We use the notation and results of Appendix A with minimal further comment. Each n -loop set contains diagrams with $2n$ vertices but no counterterm or mass insertions, plus other diagrams of the same order containing counterterm vertex insertions (as usual) but also mass vertex insertions. We describe $(n+)$ -loops, only present in Case A1, below. For the boson self-energy, each set gives rise to components of $\pi_{\mu\nu}(k)$, i.e. of $\pi(k^2)$, $\tau(k^2)$, $\rho(k^2)$.

For Case A2, the n -loop component of $\pi(k^2)$ is given by

$$\pi_{2n}^{(2)}(k^2) = g^{2n} [\pi_{2n,n}^{(2)}(\hbar)\epsilon^{-n} + \pi_{2n,n-1}^{(2)}(k^2, \hbar)\epsilon^{-n+1} + \dots + \pi_{2n,0}^{(2)}(k^2, \hbar)\epsilon^0] \quad (\text{A.2})$$

where $\pi_{2n,n}^{(2)}$ is independent of k^2 and real. There are parallel expressions for $\tau_{2n}^{(2)}(k^2)$ and $\sigma_{2n}^{ij(2)}(k^2)$. We define the following leading divergent parts:

$$\pi_{2nL}^{(2)} = g^{2n} \pi_{2n,n}^{(2)} \epsilon^{-n}, \quad \tau_{2nL}^{(2)} = g^{2n} \tau_{2n,n}^{(2)} \epsilon^{-n}, \quad \sigma_{2nL}^{ij(2)} = g^{2n} \sigma_{2n,n}^{ij(2)} \epsilon^{-n}, \quad (\text{A.4})$$

which are real and independent of k^2 , so that (see (A.14))

$$\hat{\pi}_{2nL}^{(2)} = \pi_{2nL}^{(2)}, \quad \hat{\tau}_{2nL}^{(2)} = \tau_{2nL}^{(2)}, \quad \hat{\sigma}_{2nL}^{ij(2)} = \sigma_{2nL}^{ij(2)}. \quad (26)$$

For Case B we have the components

$$\pi_{n,n}^{(B)}(k^2) = w^{n+1} [\pi_{n,n}^{(B)} \epsilon^{-n} + \pi_{n,n-1}^{(B)}(k^2) \epsilon^{-n+1} + \dots], \quad (\text{A.5})$$

(where $\pi_{n,j}^{(B)}$ are functions of g^2) with similar expressions for $\tau_{n,n}^{(B)}(k^2)$ and $\sigma_{n,n}^{ij(B)}$, and leading divergent parts, independent of k^2 and real, analogous to those of Case A2 in (A.4); also, analogues of (26) hold.

For Case A1, the n -loop diagrams, each of which must contain an even number of fermion mass insertions, lead to components $\pi_{2n}^{(1)}(k^2)$, $\tau_{2n}^{(1)}(k^2)$, $\sigma_{2n}^{ij(1)}(k^2)$, and leading divergent parts, that are similar in form to those of their counterparts in Case A2, exemplified by (A.3), (A.4). In addition, there are diagrams containing an odd number of fermion mass insertions that generate terms of $O(g^{2n+1})$, for which there are no corresponding $2n$ -vertex-only loops. We refer to such diagrams as $(n+)$ -loop diagrams. They give the components

$$\pi_{2n+1}^{(1)}(k^2) = g^{2n+1} [\pi_{2n+1,n}^{(1)} \epsilon^{-n} + \pi_{2n+1,n-1}^{(1)}(k^2) \epsilon^{-n+1} + \dots], \quad (\text{A.7})$$

where $n \geq 1$ and $\pi_{2n+1,n}^{(1)}$ is real and independent of k^2 , plus components $\tau_{2n+1}^{(1)}(k^2)$ and $\sigma_{2n+1}^{ij(1)}(k^2)$ of similar form. Since each $(n+)$ -loop diagram may be regarded as a Case A2 n -loop into which an additional mass vertex has been inserted, and such an insertion cannot increase the degree of divergence of each of the n -loop components ($\pi_{2n}^{(2)}(k^2)$, etc.), we see that $\pi_{2n+1,n}^{(1)}$ could be zero, and similarly for τ, ρ . The leading divergent parts of the components of $\pi(k^2)$, which are real and independent of k^2 , are

$$\pi_{2nL}^{(1)} = g^{2n} \pi_{2n,n}^{(1)} \epsilon^{-n}, \quad \pi_{(2n+1)L}^{(1)} = g^{2n+1} \pi_{2n+1,n}^{(1)} \epsilon^{-n}, \quad (\text{A.8})$$

where $\pi_{(2n+1)L}^{(1)}$ might be zero, and analogues of (26) hold.

To renormalise loop by loop, we impose the condition

$$(\hbar\hat{\pi}_{qL}^{(J)} - c_{3q}^{(J)})M^2 - \hbar \sum m_i m_j \hat{\sigma}_{qL}^{ij(J)} = 0, \quad (27)$$

where the latter term is $\hbar\hat{\rho}_{qL}^{(J)}$, $J = 1, 2, B$ labels the case, and $q = 2n$ for $J = 2$, $q = n$ for $J = B$, and $q = 2n, 2n+1$ for $J = 1$. The quantities in (27) are all independent of k^2 . Since g in Cases A1, A2, and $w = \hbar^{-1}$ in Case B, are variable expansion parameters, and m_i, m_j are given by (6), (7), (8), we must have

$$M = Bg \text{ (Case A1), } \quad M = Bg^2 \text{ (Case A2), } \quad M = B\hbar^{-1} = Bw \text{ (Case B).} \quad (28)$$

With $\hbar = 1$ finally, our three cases have fermion and boson masses of $O(1)$, $O(g)$ and $O(g^2)$.

The condition (27) determines c_3 . We take the counterterm parameter $c_{3q}^{(J)}$ to have the form of $\hbar^{-1}\pi_{qL}^{(J)}$, and all other potential components of c_3 to be zero. In contrast to the usual complete cancellation of all divergences in QED and QCD (e.g. with minimal subtraction, MS), we effect only an order by order cancellation of the most divergent part of each component of $\pi(M^2)M^2 - \rho(M^2)$, and do *not* make any cancellation of less divergent quantities.

In Case A2 at one loop, we have, from (22), (27),

$$e_1 = (1 + \hbar\hat{\pi}_{2L}^{(2)} - c_{3(2)}^{(2)})M^2 - \hbar\hat{\rho}_{2L}^{(2)} + O(\epsilon^0) \quad (29)$$

$$= O(\epsilon^0), \quad (30)$$

while, from (23), (27),

$$(\Omega_3^{-1})_1 = 1 + \hbar\hat{\pi}_{2L}^{(2)} - c_{3(2)}^{(2)} + O(\epsilon^0) \quad (31)$$

$$= O(\epsilon^{-1}), \quad (32)$$

and, since $\hat{\pi}_2^{(2)'}$, $\hat{\rho}_2^{(2)'}$, etc. are $O(\epsilon^0)$,

$$R_1(k^2) = O(\epsilon^0). \quad (33)$$

Accordingly, to one loop in Case A2 we have

$$\Omega_3 e = O(\epsilon), \quad (34)$$

$$\Omega_3 R(k^2) = O(\epsilon) \quad (35)$$

and, from (22),

$$d = \Omega_3^{-1}[k^2 - M^2 + O(\epsilon)]. \quad (36)$$

In Appendix B.1 we show that going to two loops in Case A2 gives (36) again. The result extends to N loops, N arbitrary.

In Case B at one loop, with (22), (23), (27) and since $\hat{\pi}_{1L}^{(B)}$ is $O(w^2 g^2 \epsilon^{-1})$ and $\hat{\rho}_{1L}^{(B)}$ is $O(w^4 g^2 \epsilon^{-1})$, we see that

$$e_1 = (1 + w^{-1} \pi_{1L}^{(B)} - c_3^{(B)}) M^2 - w^{-1} \rho_{1L}^{(B)} = O(\epsilon), \quad (37)$$

and Ω^{-1} is $O(\epsilon^{-1})$, so that we obtain (34), (35), (36) again. The result extends to N loops. In Appendix B.1 we obtain (36) in Case A1, proceeding from 1-loop to (1+)-loops, 2-loops, (2+)-loops and so on.

We see from (17), (19) and (36) that Ω_3 is the renormalisation factor that multiplies the renormalised propagator $i\mathcal{D}_{\mu\nu}^R$ obtained below. We also see, using (A.3), (A.5), (A.7), that $(\Omega_3^{-1})_n$ and $(\Omega_3^{-1})_{n+}$ are $O(\epsilon^{-n})$, while $R_n(k^2)$, $R_{(n+)}(k^2)$ are $O(\epsilon^{-n+1})$.

The BRS Ward identities of (1) impose no constraint on Z_4 (the ξ dependence being unphysical), allowing c_4 to be chosen arbitrarily. To renormalise $Q(k^2)$, we take the components of c_4 to be

$$c_{4q}^{(J)} = -\xi[\lambda^{-1}(\hbar\pi_{qL}^{(J)} - c_{3q}^{(J)}) + \hbar\tau_{qL}^{(J)}], \quad (38)$$

where λ is arbitrary. Working up to N -loops or $(N+)$ -loops, we see that in each case the numerator and denominator of $Q(k^2)$ are dominated, as $\epsilon \rightarrow 0$, by the $O(\epsilon^{-N})$ terms in $\pi_{UL}^{(J)}$, $c_{3U}^{(J)}$, $\tau_{UL}^{(J)}$, $\sigma_{UL}^{ij(J)}$, where $U = 2N, 2N+1$. Using (27), we obtain

$$\begin{aligned} Q(k^2)_U^{(J)} &= \frac{(1 - \lambda^{-1})(\hbar\pi_{UL}^{(J)} - c_{3U}^{(J)})}{(k^2\lambda^{-1} - M^2)(\hbar\pi_{UL}^{(J)} - c_{3U}^{(J)})} + O(\epsilon) \\ &= \frac{\lambda - 1}{k^2 - \lambda M^2} + O(\epsilon), \end{aligned} \quad (39)$$

which, as $\epsilon \rightarrow 0$, is independent of J , N and ξ . The gauge parameter ξ has been replaced by an unrelated, arbitrary parameter λ , which we refer to as the pseudo-gauge parameter.

From (17), (19), (23), (36) and (39) we obtain, as $\epsilon \rightarrow 0$, the renormalised propagator

$$i\mathcal{D}_{\mu\nu}^{R(loop)}(k) = \frac{-i\hbar}{k^2 - M^2 + i\epsilon'} \left[g_{\mu\nu} + (\lambda - 1) \frac{k_\mu k_\nu}{k^2 - \lambda M^2 + i\epsilon'} \right] \quad (40)$$

(where we have re-inserted $i\epsilon'$) and the renormalisation factor

$$\Omega_3 = [1 + \hbar\hat{\pi} - c_3 + M^2\hbar\hat{\pi}' - \hbar\hat{\rho}']^{-1}, \quad (41)$$

which are correct to arbitrary order. The propagator (40) is independent of ξ .

3.2 Complex mass, $m_{AR} = \hbar\mathcal{M}$

For a complex mass $\hbar\mathcal{M}$ we cannot renormalise loop by loop, since (27) cannot hold if M^2 is replaced by \mathcal{M}^2 . Instead, we renormalise order by order. With \mathcal{M}^2 of the form (20) and M given by (28), we take, for simplicity, δ to be of $O(g^0\hbar^0)$, so that \mathcal{M}^2 is $O(g^2)$, $O(g^4)$ or $O(w^2)$ in Cases A1, A2 or B. The treatment that follows can be modified to accommodate $\delta = \delta_0 + \delta_1 g + \dots$ in A1 or A2 (at any given order $O(g^2)$, a term $\delta_\sigma g^\sigma$ leads to $O(g^n)$ terms that are less divergent than the dominant divergent components) or $\delta = \delta_0 + \delta_1 w + \dots$ in Case B.

Again we determine c_3 by (27), and (28) holds. Up to $O(g^2)$ in A1, A2 and $O(w)$ in B, we have

$$e_{g^2}^{(1)} = B^2 g^2 (1 - i\delta), \quad e_{g^2}^{(2)} = 0, \quad e_w^{(B)} = 0, \quad (42)$$

which are $O(\epsilon^0)$ (there is no need to differentiate B into $B^{(1)}$, $B^{(2)}$, $B^{(B)}$, or δ , β_1 , β_2 , β_A similarly, in what follows), and, with $\hbar = 1$ in A1, A2,

$$(\Omega_3^{-1})_{g^2}^{(1)} = 1 + g^2 \tilde{\pi}_{2,1}^{(1)} \epsilon^{-1} + O(\epsilon^0), \quad (43)$$

$$(\Omega_3^{-1})_{g^2}^{(2)} = 1 + g^2 \tilde{\pi}_{2,1}^{(2)} \epsilon^{-1} + O(\epsilon^0), \quad (44)$$

$$(\Omega_3^{-1})_{\hbar}^{(B)} = 1 + w \tilde{\pi}_{1,1}^{(B)} \epsilon^{-1} + O(\epsilon^0), \quad (45)$$

which are $O(\epsilon^{-1})$. Here we have written

$$\hat{\pi}_{q,n}^{(J)} - \hbar^{-1} c_{3(q)}^{(J)} = \tilde{\pi}_{q,n}^{(J)}. \quad (46)$$

In addition, we see from (24) that $R_{g^2}^{(1)}(k^2)$, $R_{g^2}^{(2)}(k^2)$ and $R_{\hbar}^{(B)}(k^2)$ are each $O(\epsilon^0)$. Consequently, we obtain (34), (35) and (36) to $O(g^2)$, $O(w)$ in Cases A1, A2 and B.

In Appendix B we continue the argument to $O(g^4)$, $O(g^6)$ in Case A2, $O(w^2)$, $O(w^3)$ in Case B and $O(g^3)$, $O(g^4)$ in Case A1. It is evident that we can continue to an arbitrarily high order. The procedure “cuts across

loops". At lowest order it links the "zero loop" \mathcal{M}^2 to a one-loop term. Up to $O(g^3)$ in Case A1, it involves zero-loop, one-simple-loop and one-loop-with-mass-insertion contributions. In A2 up to $O(g^4)$ and B up to $O(w^2)$, it links zero-loop and two-loop terms, and up to $O(g^6)$, $O(w^3)$ links one-loop and three-loop contributions.

The similarity in the renormalisation of Cases A2 and B shows that, in a sense, w is equivalent to g^2 . If we had chosen the masses m_1, m_2, m_A in Case B to be of the form $m = \beta w^{1/2}$, then renormalisation in Case B would proceed as it does in Case A1, i.e. up to $w, w^{3/2}, \dots$ etc.

With the components of c_4 given by (38), we expand the numerator and denominator of $Q(k^2)$ up to $O(g^{2N})$ or $O(g^{2N+1})$, in Cases A1 and A2, and to $O(w^N)$ in Case B, with N arbitrarily large. The highest divergence, of $O(\epsilon^{-N})$, is in $\pi(k^2)$, $\hat{\pi}_L$, c_3 to these orders, while the highest divergence in ρ to these orders is $O(\epsilon^{-N+k})$, because of the factors $m_i m_j$, which are of $O(g^2)$, $O(g^4)$, $O(w^2)$ in the three cases, in ρ . Consequently, to $O(g^{2N})$, $O(g^{2N+1})$, $O(w^N)$, Q is

$$Q(k^2) = \frac{(1 - \lambda^{-1})\tilde{\pi}_{UN}^{(J)}[1 + O(\epsilon)]}{k^2 \lambda^{-1} \tilde{\pi}_{UN}^{(J)}[1 + O(\epsilon)]} \quad (47)$$

where $U = 2N, 2N + 1$. In the limit $\epsilon \rightarrow 0$ we obtain (cf. (39))

$$Q(k^2) = \frac{\lambda - 1}{k^2}, \quad (48)$$

which is independent of N and ξ , and the renormalised propagator is

$$i\mathcal{D}_{\mu\nu}^{R(order)}(k) = \frac{-i\hbar}{k^2 - \mathcal{M}^2 + i\epsilon'} \left[g_{\mu\nu} + (\lambda - 1) \frac{k_\mu k_\nu}{k^2 + i\epsilon'} \right]. \quad (49)$$

The renormalisation factor Ω_3 is again given by (41).

3.3 Choice of λ

The propagators (40), (49) contain unphysical poles at $k^2 = \lambda M^2$, $k^2 = 0$, as does the unrenormalised propagator (13) at $k^2 = \xi \mu^2$. For a stable boson ($\mathcal{M} = M < 2m_1$), we may take the limit $\lambda \rightarrow \infty$ to obtain

$$i\mathcal{D}_{\mu\nu}^{(R)(\infty)}(k) = \frac{-i\hbar}{k^2 - M^2 + i\epsilon'} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \right], \quad (50)$$

which is the form usually expected for a massive vector boson. In theories with SSB (or in (13)), this propagator would result from unitary gauge

($\xi \rightarrow \infty$), but here we can take $\lambda \rightarrow \infty$ for any value of ξ , the unitary pseudo-gauge. However, we cannot take $\lambda \rightarrow \infty$ in (49) for an unstable boson.

The choice of $\lambda = 1$ in c_4 results in the same propagator for either stable or unstable bosons, viz.,

$$i\mathcal{D}_{\mu\nu}^{R(1)}(k) = \frac{-i\hbar \cdot g_{\mu\nu}}{k^2 - \mathcal{M}^2 + i\epsilon'}. \quad (51)$$

There is no discontinuity in form passing from $M < 2m_1$ to $M > 2m_1$. We can choose $\lambda = 1$ for any value of ξ . In a sense, ξ in (13) has been renormalised to λ , $\lambda = 1$, or in (50), for a stable boson only, to λ , $\lambda \rightarrow \infty$.

The Ward identities with non-zero boson mass imply that it is the spin-0 part of the virtual $f\bar{f}$ self-energy loop that acts as an effective longitudinal degree of freedom [40].

4 Renormalisation of the Fermion Propagators

The full improper propagator for the ψ or η fermion is given by the series

$$i\mathcal{S}_F(\not{p}) = i(\not{p} - \kappa)(1 + u + u^2 + \cdots), \quad (52)$$

where

$$u = [\Sigma(\not{p}) - \kappa + s\not{p}](\not{p} - \kappa)^{-1}, \quad (53)$$

in which $\Sigma(\not{p})$ is given by the self-energy $-i\Sigma(\not{p})$, the first $\kappa = m\hbar^{-1}$ is from $-\hbar^{-1}\chi$ in $\hbar^{-1}\mathcal{L}_1$, and s is a counterterm parameter given below. Here m , Σ , κ , s are m_j , Σ_j , κ_j , s_j , with $j = 1$ for ψ , $j = 2$ for η . We often omit these subscripts in what follows. We write

$$\Sigma(\not{p}) = -a(p^2)(\not{p}) + \kappa b(p^2), \quad (54)$$

where

$$a(p^2) = a_{(1)}(p^2) + a_{(5)}(p^2)\gamma^5, \quad b(p^2) = b_{(1)}(p^2) + b_{(5)}(p^2)\gamma^5 \quad (55)$$

(we note that $b_{(5)}(p^2) = 0$ at one loop), more explicitly, we have $a_{j(1)}$, $a_{j(5)}$, $b_{j(1)}$, $b_{j(5)}$. Similarly we write

$$s = s_{(1)} + s_{(5)}\gamma^5, \quad (56)$$

where, from (1),

$$s_{1(1)} = \frac{1}{2}(c_{2L} + c_{2R}), \quad s_{1(5)} = \frac{1}{2}(c_{2L} - c_{2R}), \quad (57)$$

$$s_{2(1)} = \frac{1}{2}(c_{6L} + c_{6R}), \quad s_{2(5)} = \frac{1}{2}(c_{6L} - c_{6R}). \quad (58)$$

We take

$$f \neq \pm 1, 0, \quad (59)$$

$f \neq \pm 1$ so that at one loop the mass-regenerating terms $b_1(p^2)$, $b_2(p^2)$ do not vanish, and $f \neq 0$ so that at one loop the boson mass (section 3) does not vanish.

In Case A1 or A2 we can sum (52) correctly to any given $O(g^n)$, and in Case B, correctly to any given $O(w^n)$ (see the discussion of boson propagator series summation in section 3), to obtain, using (53), (54),

$$\begin{aligned} i\mathcal{S}_F(\not{p}) &= i(\not{p} - \kappa)^{-1}(1 - u)^{-1} \\ &= i[(1 - u)(\not{p} - \kappa)]^{-1} \end{aligned} \quad (60)$$

$$= i[\not{p} - \kappa - (\Sigma - \kappa + s\not{p})]^{-1} \quad (61)$$

$$= i\{[1 + a(p^2) - s]\not{p} - \kappa b(p^2)\}^{-1}, \quad (62)$$

where we have used $A^{-1}B^{-1} = (BA)^{-1}$ to reach (60) (u contains γ^5). The expected cancellation of $-\kappa$, κ from \mathcal{L}_0 , \mathcal{L}_1 may be seen in (61); however, the term κ in the propagator $i(\not{p} - \kappa)^{-1}$ has been responsible for the generation of the mass term $\kappa b(p^2)$, which leads below to the renormalised mass m .

We expand $a(p^2)$ about $\not{p} = \kappa$ to obtain

$$\begin{aligned} a(p^2) &= \hat{a} + (p^2 - \kappa^2)\hat{a}' + \frac{1}{2}(p^2 - \kappa^2)^2\hat{a}'' + \dots \\ &= \hat{a} + 2\kappa\hat{a}'(\not{p} - \kappa) + F(\not{p})(\not{p} - \kappa)^2, \end{aligned} \quad (63)$$

and expand $b(p^2)$ similarly; where $\hat{a}, \hat{b}, \hat{a}', \dots$ are defined in Appendix A (following (A.13)) and $F(\not{p})$ depends on observing the order of the factors shown in (63). We expand about $\not{p} = \kappa$, i.e. about $\not{p} = \kappa_1$, $\not{p} = \kappa_2$, because the renormalised masses m_{1R} , m_{2R} must equal the initial masses m_1 , m_2 , since each fermion is stable, so that the S -matrix reduction formula must operate in the usual way. Using (63), the denominator of $i\mathcal{S}_F(\not{p})$ in (62) becomes

$$\begin{aligned} d_f &= [1 + a(p^2) - s]\not{p} - \kappa b(p^2) \\ &= E + \Omega_2^{-1}(\not{p} - \kappa) + T(\not{p})(\not{p} - \kappa)^2 \end{aligned} \quad (64)$$

where, with $\hat{a} = a(\kappa^2)$, $\hat{b} = b(\kappa^2)$,

$$E = (1 + \hat{a} - s - \hat{b})\kappa, \quad (65)$$

$$\Omega_2 = [1 + \hat{a} - s + 2\kappa^2(\hat{a}' - \hat{b}')]^{-1} \quad (66)$$

(the factor analogous to Ω_2 in QED is commonly written as Z_2).

We renormalise loop by loop, as was done in section 3.1 for the boson. In Cases A1, A2 and B the leading divergent parts of \hat{a} are, from (A.10) and the analogues of (A.14),

$$\left. \begin{aligned} \hat{a}_{j,2nL}^{(1)} &= g^{2n} a_{j,2n,n}^{(1)} \epsilon^{-n}, & \hat{a}_{j,(2n+1)L}^{(1)} &= g^{2n+1} a_{j,2n+1,n}^{(1)} \epsilon^{-n}, \\ \hat{a}_{j,2nL}^{(2)} &= g^{2n} a_{j,2n,n}^{(2)} \epsilon^{-n}, & \hat{a}_{j,nL}^{(B)} &= w^{n+1} a_{j,n,n}^{(B)} \epsilon^{-n}, \end{aligned} \right\} \quad (67)$$

and similarly for \hat{b} . It is convenient to introduce the notation, similar to that in (27), of $\hat{a}_{jqL}^{(J)}$, $\hat{b}_{jqL}^{(J)}$, where $J = 1, 2, B$ labels Cases A1, A2, B, and $q = 2n$ for $J = 2, B$, while $q = 2n, 2n+1$ for $J = 1$. From Appendix A (see (A.9), (A.12), (A.13)) we see that $\hat{a}_{jq}^{'(J)}$, $\hat{b}_{jq}^{'(J)}$, $\hat{a}_{jq}^{''(J)}$, $\hat{b}_{jq}^{''(J)}$, \dots are $O(\epsilon^{-n+1})$. From (63) and the similar expansion of $b(p^2)$, and (65), (66), it follows that $T(\not{p})$ only involves $\hat{a}', \hat{b}', \dots$, so that, up to $n, (n+)$ -loops,

$$T(\not{p})_{n(n+)} = O(\epsilon^{-n+1}). \quad (68)$$

We impose the condition (cf. (27)) that

$$\hat{a}_{jqL}^{(J)} - s_{jq}^{(J)} - b_{jqL}^{(J)} = 0, \quad (69)$$

which is to hold order by order, and determines the components of the counterterm parameter s , viz.,

$$s_{jq}^{(J)} = g^q (a_{j,q,n}^{(J)} - b_{j,q,n}^{(J)}) \epsilon^{-n}. \quad (70)$$

All other potential components of κ are taken to be zero. We write

$$s_{jq}^{(J)} = s_{jq(1)}^{(J)} + s_{jq(5)}^{(J)} \gamma^5. \quad (71)$$

The counterterm parameters $c_{2L}^{(J)}$, $c_{2R}^{(J)}$, $c_{6L}^{(J)}$, $c_{6R}^{(J)}$ are then given by (57), (58), (70) and (71).

A discussion like that in section 3.1 for the loop by loop renormalisation of the boson propagator and the use of (64), (65), (66), (68) and (69) shows that, up to $n(n+)$ -loops,

$$E = O(\epsilon^{-n+1}), \quad \Omega_2^{-1} = O(\epsilon^{-n}), \quad (72)$$

$$\Omega_2 E = O(\epsilon), \quad \Omega_2 T(\not{p}) = O(\epsilon), \quad (73)$$

so that (64) gives

$$d_{fj} = \Omega_{2j}^{-1} [\not{p} - \kappa_j + O(\epsilon)] \quad (74)$$

for the denominators of the ψ and η propagators, with

$$\Omega_{2j}^{(J)} = [1 + \hat{a}_j^{(J)} - s_j^{(J)} + 2\kappa_j^2(\hat{a}_j^{'(J)} - \hat{b}_j^{'(J)})]^{-1}, \quad (75)$$

which contains γ^5 . From (62), (64) and (74), we obtain, using $(AB)^{-1} = B^{-1}A^{-1}$,

$$i\mathcal{S}_{Fj}(\not{p}) = \frac{i}{\not{p} - \kappa_j + i\epsilon' + O(\epsilon)} \cdot \Omega_{2j}^{(J)}. \quad (76)$$

In the limit $\epsilon \rightarrow 0$, we obtain the renormalised propagators

$$i\mathcal{S}_{F\psi}^R(\not{p}) = \frac{i}{\not{p} - \kappa_1 + i\epsilon'}, \quad i\mathcal{S}_{F\eta}^R(\not{p}) = \frac{i}{\not{p} - \kappa_2 + i\epsilon'}. \quad (77)$$

The renormalisation factors $\Omega_{2\psi}^{(J)} = \Omega_{21}^{(J)}$, $\Omega_{2\eta}^{(J)} = \Omega_{22}^{(J)}$ stand to the right of these propagators. We can interpret the result that $\Omega_{2\psi}^{(J)}$, $\Omega_{2\eta}^{(J)}$ contain γ^5 by saying that, similarly to the situation in standard electroweak theory [37], the left and right components of ψ, η are renormalised differently, since, dropping j, J , we can write $\Omega_2 = \tilde{\Omega}_2(1 + \Lambda\gamma^5)$, with $\tilde{\Omega}_2, \Lambda$ free of γ^5 , so that (76) becomes, as $\epsilon \rightarrow 0$,

$$i\mathcal{S}_F^R(\not{p}) = \tilde{\Omega}_2 \left[(1 + \Lambda) \frac{i}{\not{p} - \kappa + i\epsilon'} a_R + (1 - \Lambda) \frac{i}{\not{p} - \kappa + i\epsilon'} a_L \right], \quad (78)$$

where $a_R = \frac{1}{2}(1 + \gamma^5)$, $a_L = \frac{1}{2}(1 - \gamma^5)$.

5 The Vertices

It is straightforward to write out the sequence of all possible skeleton diagrams, up to any given order $O(g^{2n})$, for a given matrix element S_{f_i} ; with the definition that, for this model, a skeleton diagram contains no self-energy insertions and no vertex loops, which means no subgraphs with only two “external” lines and none, except for point vertices, with three “external” lines. We then replace each propagator and point vertex by the full improper propagator and full vertex part, each taken to an arbitrarily high order. We assume that each such propagator self-energy and vertex part

comprises unambiguous integrals in $d = 4 - 2\epsilon$ dimensions, and that we can assign the counterterms order by order in such a way as to effect the partial cancellation of the divergences in π and $a(p^2)$ that we have made above, at each order $O(g^{2n})$ or $O(w^n)$. We gloss over the problem of overlapping divergences [38], treatable by standard methods [19]. The BPHZ renormalisation method [38] can be used to effect an explicit renormalisation of the model.

Proceeding from these assumptions, we see that when the full propagators and vertex parts are inserted at each $\psi\psi A$ vertex in a skeleton diagram, we obtain the product

$$P = \dots i\mathcal{S}_F^{(R)} \Omega_2(-ig\hbar^{-1})\gamma^\mu[Z_1 + fZ_{1(5)}\gamma^5 + V(p, p')]\mathcal{S}_F^R(p')\Omega_2\Omega_3^{1/2}i\mathcal{D}_{\mu\nu}^R(k)\dots, \quad (79)$$

in which the Z_1 , $Z_{1(5)}$ terms come from (1) and V is the vertex part $V(p, p') = V_{(1)}(p, p') + V_{(5)}(p, p')\gamma^5$. The interaction vertex is

$$I_V = \Omega_3^{1/2}\Omega_2\{1 - c_1 + V_{(1)}(p, p') - [f(1 - c_{1(5)}) + V_{(5)}(p, p')]\gamma^5\}(-ig\hbar^{-1}\gamma^\mu). \quad (80)$$

We define the leading divergent parts

$$\Omega_{3L}^{-1} = 1 + \hbar\tilde{\pi}_L - c_3 \quad (81)$$

$$\Omega_{2L}^{-1} = 1 + a_L - s \quad (82)$$

$$= (\Omega_{2L}^{-1})_{(1)} + (\Omega_{2L}^{-1})_{(5)}\gamma^5, \quad (83)$$

in which we have suppressed the j, J labels carried by $\tilde{\pi}_L$, a_L (of the forms given by (A.15), (A.16), s , Ω_{3L}^{-1} and Ω_{2L}^{-1}). From (41), (75) and the analysis of orders $(g^n\epsilon^{-j})$ given in sections 3 and 4,

$$\Omega_3^{-1} = \Omega_{3L}^{-1}[1 + O(\epsilon)], \quad \Omega_2^{-1} = \Omega_{2L}^{-1}[1 + O(\epsilon)]. \quad (84)$$

Similarly we obtain

$$V_{(1)}(p, p') = V_{L(1)}[1 + O(\epsilon)], \quad V_{(5)}(p, p') = V_{L(5)}[1 + O(\epsilon)], \quad (85)$$

where the leading divergent parts $V_{L(1)}$, $V_{L(5)}$, of forms given by (A.19), (A.20), are independent of (p, p') (we have suppressed j, J labels).

We write (80) as

$$I_V = \Omega_{3L}^{1/2}\Omega_{2L}(\alpha + \beta\gamma^5)(1 - f\gamma^5)[1 + O(\epsilon)](-ig\hbar^{-1}\gamma^\mu), \quad (86)$$

where

$$\alpha = (1 - f^2)^{-1} \{1 - c_1 + V_{L(1)} - f[f(1 - c_{1(5)}) + V_{L(5)}]\}, \quad (87)$$

$$\beta = (1 - f^2)^{-1} \{f(1 - c_1 + V_{L(1)}) - [f(1 - c_{1(5)}) + V_{L(5)}]\}. \quad (88)$$

We choose α, β to be, i.e. we choose the values of $c_1, c_{1(5)}$ to be such that,

$$\alpha = (\Omega_{2L}^{-1})_{(1)} \Omega_{3L}^{-1/2}, \quad \beta = (\Omega_{2L}^{-1})_{(5)} \Omega_{3L}^{-1/2}. \quad (89)$$

Then (86) reduces to

$$I_V = (1 - f\gamma^5)[1 + O(\epsilon)](-ig\hbar^{-1}\gamma^\mu), \quad (90)$$

so that, taking $\epsilon \rightarrow 0$, the renormalised $\psi\psi A$ coupling at vertices is

$$-ig_R\hbar^{-1}\Gamma^\mu = -ig\hbar^{-1}\gamma^\mu(1 + f\gamma^5), \quad (91)$$

with $g_R = g$, which is the original unrenormalised coupling in \mathcal{L}_{eff} , (1), divided by \hbar .

In Appendix C we find the values of $c_1, c_{1(5)}$ that give this result, and mention the similar treatments to obtain $c_1, c_{1(5)}$ in Cases B and A1.

In a similar way, we can obtain the values of $c_5, c_{5(5)}$ that renormalise the $\eta\eta A$ coupling to its original form, $+ig\bar{\eta}\gamma^\mu(1 + f\gamma^5)\eta A_\mu$, in each of Cases A1, A2 and B.

6 Summary and Discussion

We have demonstrated that a theory in which the action possesses chiral and gauge symmetries has a *perturbative* solution with nonzero gauge boson masses (17), (19), (21), (40) and fermion masses (62), (64), (77). The axial vector coupling gf must be nonzero. The essential mechanism is: *the tree-level fermion masses of \mathcal{L}_0 cancel in iS_F but leave an indirect, non-vanishing effect in Σ (54).*

A full Lagrangian density \mathcal{L} does not by itself imply a unique spectrum. Starting from the chiral- and BRS-invariant \mathcal{L}_{eff} , (1), we choose an \mathcal{L}_0 that contains a massive boson and massive fermions, then renormalise in a way that dynamically regenerates via quantum corrections the same fermion masses and the same boson mass or, in the unstable case, a complex boson mass. This choice of a chirally-asymmetric \mathcal{L}_0 and consequent

asymmetric states is analogous to the choice of asymmetric rather than symmetric solutions in Schwinger-Dyson equations [9,10]. As usual in perturbation theory, our choice of \mathcal{L}_0 also breaks gauge symmetry: the partial-action $S_0 = \int d^4x \mathcal{L}_0$ does not possess $U(1)$ BRS-invariance. However, S -matrix elements are gauge-invariant.

On renormalisation to one loop, the model contains a massive gauge boson and two massive fermions, and we sketch a renormalisation procedure extending to any order. The renormalised propagators and couplings, and so S -matrix elements, are independent of the gauge parameter ξ . The model can be renormalised to all orders because the full Lagrangian \mathcal{L}_{eff} , (1), can be. No new counterterms beyond the types exhibited in (1) (in particular, no mass counterterms) should appear. In combination with causality and dispersion relations, renormalisability implies unitary high-energy behaviour. A proof of renormalisability would make use of the BRS Ward identities of (1), similar to the analogous identities of [9,40], with three important differences: our model is anomaly-free; it is solved perturbatively after the finite shift of masses from zero; and the gauge and fermion masses are *independent* of one another. This independence is a result of their *separate* renormalisations and the fact that the fermion self-energies are “hard” constant masses. Jackiw and Johnson, on the other hand, obtained an alternate solution with a “soft” fermion self-energy and finite relation between m_{AR} and $m_{1,2R}$.

Some questions remain to be explored. (1) The issue of unitarity with unstable particles is mentioned in section 1. (2) A complete treatment also requires consideration of vacuum symmetries, energy, and stability. (3) We have introduced a Dirac mass ansatz for the fermions, but Majorana masses might also be possible. These would break the residual $U(1)$ vector symmetry of fermions left after the full chiral symmetry is broken. (4) The connection between our perturbative treatment and previous non-perturbative solutions [7,8,9,10] is not fully elucidated.

The mechanism presented here is used in an electroweak theory that contains only W , Z , photon, ghost, lepton and quark fields, and in which renormalisation to one loop gives the particles their final masses [41].

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Appendix A. Definitions and Properties of $\pi_{2n}(k^2)$, etc.

We have placed powers of \hbar (dimensionless but not yet unity) in \mathcal{L}_{eff} , (1), because \hbar^{-1} is the expansion parameter in Case B, rather than g as in Cases A1, A2. When $\hbar \neq 1$, the boson propagator (13) carries a factor \hbar , and diagrams are generated by $\hbar^{-1}\mathcal{L}_1$ [37,38], as illustrated by the calculation of $\mathcal{D}_{\mu\nu}(k)$ in section 3.

That set S^{00} of diagrams which contribute to $\pi_{\mu\nu}(k)$ and contain $2n$ fermion–fermion–boson (*ffb*) vertices, but no counterterm or mass insertion vertices, generates components $\pi_{2n}^{00}(k^2)$, $\tau_{2n}^{00}(k^2)$ and $\sigma_{2n}^{ij00}(k^2)$ (see (17)) which are of order $O(w^{n+1}g^{2n})$, as is easily seen by drawing diagrams. To the set S^{00} we add, as usual, all diagrams containing fewer vertices but all possible combinations of counterterm insertions such that the order remains $O(w^{n+1}g^{2n})$, to obtain the set S^0 . In line with the results of calculations beyond one loop in QED and QCD given in the literature and the discussion of a two-loop case given by Collins [19], we assume that S^0 generates a component of $\pi^0(k^2)$ of the form

$$\pi_{2n}^0(k^2) = w^{n+1}g^{2n}[\alpha_{2n,n}\epsilon^{-n} + \alpha_{2n,n-1}(k^2)\epsilon^{-n+1} + \cdots + \alpha_{2n,n}(k^2)] \quad (\text{A.1})$$

where $w = \hbar^{-1}$ and $\alpha_{2n,n}$ is independent of k^2 and real, plus components $\tau_{2n}^0(k^2)$, $\sigma_{2n}^{ii0}(k^2)$ of $\tau^0(k^2)$, $\sigma^{ij0}(k^2)$, of the same form. Similarly, we assume that the S^0 sets for the fermion self-energies $\Sigma_1(\not{p})$, $\Sigma_2(\not{p})$ generate components of the form of

$$a_{1(2n)}^0(p^2) = w^n g^{2n}[\beta_{2n,n}\epsilon^{-n} + \beta_{2n,n-1}(p^2)\epsilon^{-n+1} + \cdots + \beta_{2n,n}(p^2)] \quad (\text{A.2})$$

for the functions $a_1^0(p^2)$, $b_1^0(p^2)$, $a_2^0(p^2)$, $b_2^0(p^2)$ (see section 4) in which the leading $w^n g^{2n}\epsilon^{-n}$ terms are again independent of k^2 and real. In the same way, we assume that the S^0 set for the vertex function $V(p, p') = V_1 + V_5\gamma^5$, defined following (79), generates expressions similar to (A.2) for V_1 and V_5 but with $\beta_{2n,n-\sigma}(p^2)$ replaced by $\beta_{2n,n-\sigma}(p, p')$.

For $\pi_{\mu\nu}(k^2)$ in Case A2, (7), we add to S^0 all diagrams containing $(2n-j)$ *ffb* vertices, mass vertex insertions (each of $O(g^2)$) and counterterm insertions such that the order remains $O(g^{2n})$. We refer to the resulting set of diagrams, $S^{(2)}$, as the n -loop diagrams for Case A2. Adding a fermion or boson mass insertion (generated by the $-\hbar^{-1}\chi$ terms in $\hbar^{-1}\mathcal{L}_1$) to a diagram

cannot increase its degree of divergence. Using (A.1), it is then easy to see that $S^{(2)}$ generates the n -loop component

$$\pi_{2n}^{(2)}(k^2) = g^{2n}[\pi_{2n,n}^{(2)}(w)\epsilon^{-n} + \pi_{2n,n-1}^{(2)}(k^2, w)\epsilon^{-n+1} + \dots + \pi_{2n,0}^{(2)}(k^2, w)] \quad (\text{A.3})$$

in which $\pi_{2n,n}^{(2)}$ is real and independent of k^2 ; and similarly generates components $\tau_{2n}^{(2)}(k^2)$, $\sigma_{2n}^{ij(2)}(k^2)$. We suppress, in the notation, the dependence of the coefficients in (A.3), and in the parallel series for $\tau_{2n}^{(2)}$, $\sigma_{2n}^{ij(2)}$, on w (in Case A2 and in Case A1 below, we may put $w = \hbar^{-1} = 1$ already at this point). We write the real, k^2 -independent leading divergent parts as

$$\pi_{2nL}^{(2)} = g^{2n}\pi_{2n,n}^{(2)}\epsilon^{-n}, \quad \tau_{2nL}^{(2)} = g^{2n}\tau_{2n,n}^{(2)}\epsilon^{-n}, \quad \sigma_{2nL}^{ij(2)} = g^{2n}\sigma_{2n,n}^{ij(2)}\epsilon^{-n}. \quad (\text{A.4})$$

In Case B, we similarly add to S^0 all diagrams of ffb vertices plus mass and counterterm insertions such that the order is $O(w^{n+1})$, and refer to the resulting set of diagrams, $S^{(B)}$, as the n -loop diagrams for Case B. We see, using (A.1), that $S^{(B)}$ generates the n -loop component

$$\pi_{n,n}^{(B)}(k^2) = w^{n+1}[\pi_{n,n}^{(B)}\epsilon^{-n} + \pi_{n,n-1}^{(B)}(k^2)\epsilon^{-n+1} + \dots], \quad (\text{A.5})$$

where $\pi_{n,n}^{(B)}$ is real and independent of k^2 , and we have suppressed (but only in the notation) the dependence of the coefficients on g^2 . $S^{(B)}$ also generates the components $\tau_n^{(B)}(k^2)$, $\sigma_n^{ij(B)}(k^2)$, with forms similar to that of $\pi_n^{(B)}(k^2)$. The leading divergent parts, $\pi_{nL}^{(B)}$ etc., are similar to those in (A.4).

In Case A1, we have, firstly, the set $S^{(1)}$ of n -loop diagrams, of ffb vertices, counterterm insertions, boson mass insertions and an even number of fermion mass insertions, such that the order is $O(g^{2n})$; which gives, as in Case A2, the component

$$\pi_{2n}^{(1)}(k^2) = g^{2n}[\pi_{2n,n}^{(1)}\epsilon^{-n} + \pi_{2n,n-1}^{(1)}(k^2)\epsilon^{-n+1} + \dots], \quad (\text{A.6})$$

together with $\tau_{2n}^{(1)}(k^2)$, $\sigma_{2n}^{ij(1)}(k^2)$, of similar forms, and leading divergent parts $\pi_{2nL}^{(1)}$, $\tau_{2nL}^{(1)}$, $\sigma_{2nL}^{ij(1)}$ of the forms of those in (A.4). In addition, there is the set of $S^{(1+)}$ diagrams that contain ffb vertices, counterterm vertices, boson mass insertions and an odd number of fermion mass vertices, such that the order is $O(g^{2n+1})$, $n \geq 1$. The set $S^{(1+)}$ does not contain diagrams that consist only of ffb vertices, so that corresponding sets S^0 , S^{00} , and S^0 components (A.1), do not exist, with the result that the first or first several

$O(\epsilon^{-n}), O(\epsilon^{-n+1}), \dots$ terms in the π, τ, σ components might be zero (the most divergent term is $O(\epsilon^{-n})$, at most, because of the argument preceding (A.3), above). Accordingly, we assume that in Case A1, the $O(g^{2n+1})$ components of $\pi(k^2)$, $\tau(k^2)$ and $\sigma^{ij}(k^2)$ are of the form of

$$\pi_{2n+1}^{(1)}(k^2) = g^{2n+1}[\pi_{2n+1,n}^{(1)}\epsilon^{-n} + \pi_{2n+1,n-1}^{(1)}(k^2)\epsilon^{-n+1} + \dots], \quad (\text{A.7})$$

where $\pi_{2n+1,n}^{(1)}$ is real and independent of k^2 , and again we define leading divergent parts

$$\begin{aligned} \pi_{(2n+1)L}^{(1)} &= g^{2n+1}\pi_{(2n+1),n}^{(1)}\epsilon^{-n}, & \tau_{(2n+1)L}^{(1)} &= g^{2n+1}\tau_{(2n+1),n}^{(1)}\epsilon^{-n}, \\ \sigma_{(2n+1)L}^{ij(1)} &= g^{2n+1}\sigma_{(2n+1),n}^{ij(1)}\epsilon^{-n}; \end{aligned} \quad (\text{A.8})$$

however, these $O(\epsilon^{-n})$ terms and some of the succeeding $O(\epsilon^{-n+1}), \dots$ terms in (A.7) and the similar series for $\tau_{2n+1}^{(1)}(k^2)$, $\sigma_{2n+1}^{ij(1)}(k^2)$, might be zero.

We assume that we have forms similar to those above for the self-energy component functions $a_1(p^2)$, $b_1(p^2)$ for ψ and $a_2(p^2)$, $b_2(p^2)$ for η (see section 4), e.g. for $a_1(p^2)$ we have, as developments from the S^0 -set assumption (A.2),

$$\left. \begin{aligned} a_{1,2n}^{(2)}(p^2) &= g^{2n}[a_{1,2n,n}^{(2)}\epsilon^{-n} + a_{1,2n,n-1}^{(2)}(p^2)\epsilon^{-n+1} + \dots], \\ a_{1n}^{(B)}(p^2) &= w^n[a_{1,n,n}^{(B)}\epsilon^{-n} + a_{1,n,n-1}^{(B)}(p^2)\epsilon^{-n+1} + \dots], \\ a_{1,2n}^{(1)}(p^2) &= g^{2n}[a_{1,2n,n}^{(1)}\epsilon^{-n} + \dots], \\ a_{1,2n+1}^{(1)}(p^2) &= g^{2n+1}[a_{1,2n+1,n}^{(1)}\epsilon^{-n} + \dots], \end{aligned} \right\} \quad (\text{A.9})$$

with the leading divergent parts

$$a_{1,2nL}^{(2)} = g^{2n}a_{1,2n,n}^{(2)}\epsilon^{-n}, \dots \quad (\text{A.10})$$

real and independent of p^2 .

For the general vertex part $g\hbar^{-1}\gamma^\mu[V_1(p, p') + \gamma^5 V_5(p, p')]$ at a $\bar{\psi}\psi$ -boson or $\bar{\eta}\eta$ -boson vertex, we assume similarly that each of V_1, V_5 consists of components of the forms (writing V for V_1, V_5), in Cases A1, A2 and B,

$$\left. \begin{aligned} V_{2n}^{(1)}(p, p') &= g^{2n}[V_{2n,n}^{(1)}\epsilon^{-n} + V_{2n,n-1}^{(1)}(p', p')\epsilon^{-n+1} + \dots], \\ V_{2n+1}^{(1)}(p, p') &= g^{2n+1}[V_{2n+1,n}^{(1)}\epsilon^{-n} + V_{2n+1,n-1}^{(1)}(p', p')\epsilon^{-n+1} + \dots], \\ V_{2n}^{(2)}(p, p') &= g^{2n}[V_{2n,n}^{(2)}\epsilon^{-n} + \dots], \\ V_n^{(B)}(p, p') &= w^n[V_{n,n}^{(B)}\epsilon^{-n} + \dots], \end{aligned} \right\} \quad (\text{A.11})$$

in which $V_{2n,n}^{(1)}$ etc. are real and independent of (p, p') . The leading divergent parts are defined by (A.19), (A.20), below. For $V_{2n+1}^{(1)}(p, p')$, the leading divergent part $g^{2n+1}V_{2n+1,n}^{(1)}\epsilon^{-n}$ and some of the succeeding terms might be zero.

We use the notations

$$\pi'(k^2) = \frac{\partial}{\partial k^2}\pi(k^2), \quad \pi'' = \frac{\partial}{\partial k^2}\pi', \dots, \quad (\text{A.12})$$

$$\hat{\pi} = \pi(\mathcal{M}^2), \quad \hat{\pi}' = \pi'(\mathcal{M}^2), \dots, \quad (\text{A.13})$$

and similarly for $\tau, \rho, a(p^2), b(p^2)$. From the k^2 -independence of the leading divergent parts $\pi_{2nL}^{(1)}$, etc., we see that

$$\begin{aligned} \hat{\pi}_{2nL}^{(1)} &= \pi_{2nL}^{(1)}, & \hat{\pi}_{(2n+1)L}^{(1)} &= \pi_{(2n+1)L}^{(1)}, \\ \hat{\pi}_{2nL}^{(2)} &= \pi_{2nL}^{(2)}, & \hat{\pi}_{nL}^{(B)} &= \pi_{nL}^{(B)}, \end{aligned} \quad (\text{A.14})$$

and that similar relations hold for τ and σ , and so for ρ .

At (27) we introduced the notation $\hat{\pi}_{qL}^{(J)}, c_{3q}^{(J)}, \sigma_{qL}^{(ij)(J)}$, where $J = 1, 2, B$ labels the case, and $q = 2n$ for $J = 2, B$, while $q = 2n, 2n+1$ for $J = 1$. At (46) we introduced $\tilde{\pi}_{q,n}$. It is also convenient to introduce

$$\tilde{\pi}_L^{(J)} = \pi_L^{(J)} - \hbar^{-1}c_3^{(J)} = \sum_n (\pi_{qL}^{(J)} - \hbar^{-1}c_{3(q)}^{(J)}), \quad (\text{A.15})$$

$$\hat{a}_{jL}^{(J)} = a_{jL}^{(J)} = \left\{ \begin{array}{l} \sum (g^{2n}a_{j,2nL}^{(1)} + g^{2n+1}a_{j,(2n+1)L}^{(1)})\epsilon^{-n} \\ \sum g^{2n}a_{j,2nL}^{(2)}\epsilon^{-n} \\ \sum w^n a_{j,nL}^{(B)}\epsilon^{-n} \end{array} \right\}, \quad (\text{A.16})$$

and

$$\begin{aligned} b_{jL}^{(J)} &= \sum m_i m_k \sigma_{jL}^{(ik)(J)} \\ &= \left\{ \begin{array}{l} \sum (g^{2n}b_{j,2nL}^{(1)} + g^{2n+1}b_{j,(2n+1)L}^{(1)})\epsilon^{-n} \\ \sum g^{2n}b_{j,2nL}^{(2)}\epsilon^{-n} \\ \sum w^n b_{j,nL}^{(B)}\epsilon^{-n} \end{array} \right\}, \end{aligned} \quad (\text{A.17})$$

in which

$$a_{j,2nL}^{(2)} = a_{j,2nL(1)}^{(2)} + a_{j,2nL(5)}^{(2)}\gamma^5, \quad (\text{A.18})$$

and similarly for $a_{j,2nL}^{(1)}$, $a_{j,(2n+1)L}^{(1)}$ and the other a, b quantities. The forms of $\sigma_{jL}^{(ik)(J)}$ are evident from (A.17). In a similar way, we write

$$V_{jL(1)}^{(J)} = \left\{ \begin{array}{l} \sum (g^{2n} V_{j,2n,n(1)}^{(1)} + g^{2n+1} V_{j,2n+1,n(1)}^{(1)}) \epsilon^{-n} \\ \sum g^{2n} V_{j,2n,n(1)}^{(2)} \epsilon^{-n} \\ \sum w^n V_{j,n,n(1)}^{(B)} \epsilon^{-n} \end{array} \right\}, \quad (\text{A.19})$$

and

$$V_{jL(5)}^{(J)} = \left\{ \begin{array}{l} \sum (g^{2n} V_{j,2n,n(5)}^{(1)} + g^{2n+1} V_{j,2n+1,n(5)}^{(1)}) \epsilon^{-n} \\ \sum g^{2n} V_{j,2n,n(5)}^{(2)} \epsilon^{-n} \\ \sum w^n V_{j,n,n(5)}^{(B)} \epsilon^{-n}. \end{array} \right\}. \quad (\text{A.20})$$

As discussed in the main text, some of $a_{j,(2n+1)L}^{(1)}$, $b_{j,(2n+1)L}^{(1)}$, $V_{j,2n+1,n(1)}^{(1)}$, $V_{j,2n+1,n(5)}^{(1)}$ might be zero.

Appendix B

B.1 Loop by loop renormalisation of the boson propagator

Following the renormalisation of the boson propagator to one loop in Case A2 in (29) to (36), we renormalise up to two loops. We obtain

$$\begin{aligned} e_2 &= [1 + \hbar \hat{\pi}_{2L}^{(2)} - c_{3(2)} + \hbar \hat{\pi}_{4L}^{(2)} - c_{3(4)}] M^2 \\ &\quad - \hbar (\hat{\rho}_{2L}^{(2)} + \hat{\rho}_{4L}^{(2)}) + O(\epsilon^{-1}) \end{aligned} \quad (\text{B.1})$$

$$= O(\epsilon^{-1}) \quad (\text{B.2})$$

on imposing (27), and

$$(\Omega_3^{-1})_2 = 1 + \hbar \hat{\pi}_{2L}^{(2)} - c_{3(2)}^{(2)} + \hbar \pi_{4L}^{(2)} - c_{3(4)}^{(2)} + O(\epsilon^{-1}) \quad (\text{B.3})$$

$$= O(\epsilon^{-2}). \quad (\text{B.4})$$

$R_2(k^2)$ is $O(\epsilon^{-1})$, so that (34), (35), (36) hold up to two loops. Proceeding in this way, we see that (36) holds to N loops, N arbitrarily large. A parallel treatment of Case B gives the same result, that (36) holds to N loops.

For Case A1 at one loop, the development, now in $\hat{\pi}_2^{(1)}$, $\hat{\rho}_2^{(1)}$, parallels that of Case A2, from (29), to reach (36) again. Then up to (1+)-loops, we have, using (A.8),

$$e_{1+} = (1 + \hbar\hat{\pi}_{2L}^{(1)} - c_{3(2)} + \hbar\hat{\pi}_{3L}^{(1)} - c_{3(3)}M^2 - \hbar(\hat{\rho}_{2L}^{(1)} + \hat{\rho}_{3L}^{(1)}) + O(\epsilon^0), \quad (\text{B.5})$$

$$(\Omega_3^{-1})_{1+} = 1 + \hbar\hat{\pi}_{2L}^{(1)} - c_{3(2)} + \hbar\hat{\pi}_{3L}^{(1)} - c_{3(3)} + O(\epsilon^0), \quad (\text{B.6})$$

$$R_{(1+)}(k^2) = O(\epsilon^0). \quad (\text{B.7})$$

Imposing (27), we see (using (A.8)) that, as for one loop, e_{1+} is $O(\epsilon^0)$ and $(\Omega_3^{-1})_{1+}$ is $O(\epsilon^{-1})$, whether or not $\hat{\pi}_{3L}^{(1)}$ is zero. Accordingly we obtain (36) again, to (1+)-loops in Case A1. Up to two loops, we find that (as in Case A2) e_2 is reduced from $O(\epsilon^{-2})$ to $O(\epsilon^{-1})$ by (27), while $(\Omega_3^{-1})_2$ is $O(\epsilon^{-2})$ and $R_2(k^2)$ is $O(\epsilon^{-1})$, which leads again to (36). Up to (2+)-loops, the argument is similar to that for (1+)-loops, with e reduced to $O(\epsilon^{-1})$ and Ω_3^{-1} of $O(\epsilon^{-2})$, $R(k^2)$ of $O(\epsilon^{-1})$, to give (36) again. We proceed in this way to any number of loops.

B.2 Order by order renormalisation of the boson propagator

In (42) to (45) we developed the renormalisation of the boson propagator to $O(g^2)$, $O(w)$ in Cases A1, A2 and B.

In Cases A2 and B we proceed order by order in g^2, w . Up to $O(g^4)$ in A2, $O(w^2)$ in B, we have

$$e_{g^4}^{(2)} = B^2 g^4 (1 - i\delta), \quad e_{w^2}^{(B)} = B^2 w^2 (1 - i\delta), \quad (\text{B.8})$$

which are $O(\epsilon^0)$, and

$$(\Omega_3^{-1})_{g^4}^{(2)} = 1 + g^2 \tilde{\pi}_{2,1}^{(2)} \epsilon^{-1} + g^4 \tilde{\pi}_{4,2}^{(2)} \epsilon^{-2} + O(\epsilon^{-1}, \epsilon^0), \quad (\text{B.9})$$

$$(\Omega_3^{-1})_{w^2}^{(B)} = 1 + w \tilde{\pi}_{1,1}^{(B)} \epsilon^{-1} + w^2 \tilde{\pi}_{2,2}^{(B)} \epsilon^{-2} + O(\epsilon^{-1}, \epsilon^0), \quad (\text{B.10})$$

(using the notation (46)), which are $O(\epsilon^{-2})$, while $R_{g^4}^{(2)}$, $R_{w^2}^{(B)}$ are $O(\epsilon^{-1})$. Again we obtain (36) to these orders. Similarly, up to $O(g^6)$, $O(w^3)$ we have

$$e_{g^6}^{(2)} = (1 + g^2 \tilde{\pi}_{2,1}^{(2)} \epsilon^{-1}) B^2 g^4 (1 - i\delta) - \sum (\beta_i \beta_j g^4) (g^2 \hat{\sigma}_{2,1}^{ij(2)} \epsilon^{-1}), \quad (\text{B.11})$$

$$e_{w^3}^{(B)} = (1 + w \tilde{\pi}_{1,1}^{(B)} \epsilon^{-1}) B^2 w^2 (1 - i\delta) - \sum (\beta_i \beta_j w^2) (w \hat{\sigma}_{1,1}^{ij(B)} \epsilon^{-1}), \quad (\text{B.12})$$

$$(\Omega_3^{-1})_{g6}^{(2)} = 1 + g^2 \tilde{\pi}_{2,1}^{(2)} \epsilon^{-1} + g^4 \tilde{\pi}_{4,2}^{(2)} \epsilon^{-2} + g^6 \tilde{\pi}_{6,3}^{(2)} \epsilon^{-3} + O(\epsilon^{-2}, \epsilon^{-1}, \epsilon^0), \quad (\text{B.13})$$

$$(\Omega_3^{-1})_{w3}^{(B)} = 1 + w \tilde{\pi}_{1,1}^{(B)} \epsilon^{-1} + w^2 \tilde{\pi}_{2,2}^{(B)} \epsilon^{-2} + w^3 \tilde{\pi}_{3,3}^{(B)} \epsilon^{-3} + O(\epsilon^{-2}, \epsilon^{-1}, \epsilon^0), \quad (\text{B.14})$$

and $R_{g6}^{(2)}, R_{w3}^{(B)}$ are $O(\epsilon^{-2})$. Again we obtain (36).

For Case A1, the treatment up to $O(g^2)$ is given in (42) to (45). Up to $O(g^3)$ we have

$$e_{g3}^{(1)} = e_{g2}^{(1)} = B^2 g^2 (1 - i\delta), \quad (\text{B.15})$$

$$(\Omega_3^{-1})_{g3} = 1 + g^2 \tilde{\pi}_{2,1}^{(1)} \epsilon^{-1} + g^3 \tilde{\pi}_{3,1}^{(1)} \epsilon^{-1} + O(\epsilon^0) \quad (\text{B.16})$$

and $R_{g2}^{(1)}, R_{g3}^{(1)}$ are $O(\epsilon^0)$, so that we obtain (34), (35) and (36) to this order. Note that going to $O(g^{2n+1})$ does not increase the degree of divergence from that at $O(g^{2n})$ (see (A.6), (A.7)). Up to $O(g^4)$ we have

$$e_{g4}^{(1)} = (1 + g^2 \tilde{\pi}_{2,1}^{(1)} \epsilon^{-1}) B^2 g^2 (1 - i\delta) - \sum (\beta_i \beta_j g^2) (g^2 \hat{\sigma}_{2,1}^{ij(1)} \epsilon^{-1}) + O(\epsilon^0), \quad (\text{B.17})$$

$$(\Omega_3^{-1})_{g4}^{(1)} = 1 + g^2 \tilde{\pi}_{2,1}^{(1)} \epsilon^{-1} + g^3 \tilde{\pi}_{3,1}^{(1)} \epsilon^{-1} + g^4 \tilde{\pi}_{4,2}^{(1)} \epsilon^{-2} + O(\epsilon^{-1}, \epsilon^0) \quad (\text{B.18})$$

and $R_{g4}^{(1)}$ is $O(\epsilon^{-1})$. We obtain (36) again. We continue to $O(g^5)$, and so on.

Appendix C. Counterterm parameters for the $\psi\psi A$ vertex

To find the values of $c_1, c_{1(5)}$ that give the result (91), we write (89) as

$$\frac{\alpha^2}{\Omega_{3L}^{-1}[(\Omega_{2L}^{-1})_{(1)}]^2} = 1, \quad \frac{\beta^2}{\Omega_{3L}^{-1}[(\Omega_{2L}^{-1})_{(5)}]^2} = 1, \quad (\text{C.1})$$

with the positive square root understood. In Case A2, we write

$$\alpha = 1 + \sum \alpha_{2n} g^{2n} \epsilon^{-n}, \quad \beta = \sum \beta_{2n} g^{2n} \epsilon^{-n}, \quad (\text{C.2})$$

$$\Omega_{3L}^{-1} = 1 + \sum p_{2n} g^{2n} \epsilon^{-n}, \quad \Omega_{2L}^{-1} = 1 + \sum (r_{2n} + t_{2n} \gamma^5) g^{2n} \epsilon^{-n}, \quad (\text{C.3})$$

where

$$p_{2n} = \pi_{1,2n,n}^{(2)}, \quad (\text{C.4})$$

$$r_{2n} + t_{2n} \gamma^5 = a_{1,2n,n}^{(2)} - s_{1,2n}^{(2)} = b_{1,2n,n}^{(2)}. \quad (\text{C.5})$$

The first equation in (C.1) is then

$$\frac{1 + 2\alpha_2 g^2 \epsilon^{-1} + (2\alpha_4 + \alpha_2^2) g^4 \epsilon^{-2} + (2\alpha_6 + 2\alpha_4 \alpha_2) g^6 \epsilon^{-3} + \dots}{1 + (p_2 + 2r_2) g^2 \epsilon^{-1} + (p_4 + 2p_2 r_2 + 2r_4 + r_2^2) g^4 \epsilon^{-2} + \dots} = 1. \quad (\text{C.6})$$

Since

$$\left[\frac{1 + \dots + P g^{2N} \epsilon^{-N}}{1 + \dots + Q g^{2N} \epsilon^{-N}} \right]_{O(g^{2N})} = \frac{1 + \dots P g^{2N} \epsilon^{-N}}{1 + \dots Q g^{2N} \epsilon^{-N}} + O(g^{2N+2}), \quad (\text{C.7})$$

where the $O(g^{2N})$ subscript denotes “up to $O(g^{2N})$ ”, we can satisfy (C.6) order by order, correctly to any given order $O(g^{2N})$, by making the numerator and denominator identical order by order, i.e. by taking

$$\left. \begin{aligned} \alpha_2 &= \frac{1}{2} p_2 + r_2, \\ \alpha_4 &= \frac{1}{2} (p_4 + p_2 r_2 + 2r_4 - \frac{1}{4} p_2^2), \end{aligned} \right\} \quad (\text{C.8})$$

(where we have used the expression for α_2 in writing α_4) and so on. In a similar way, the second equation in (C.1) leads to

$$\left. \begin{aligned} \beta_2 &= t_2, \\ \beta_4 &= t_4 + \frac{1}{2} t_2 p_2, \end{aligned} \right\} \quad (\text{C.9})$$

and so on. Then from (87), (88), (C.2), (C.8), (C.9) we obtain

$$\begin{aligned} c_1 &= V_{L(1)} + 1 - \alpha + f\beta \\ &= V_{L(1)} + (-\frac{1}{2} p_2 - r_2 + f t_2) g^2 \epsilon^{-1} \\ &\quad + [-\frac{1}{2} p_4 - \frac{1}{2} p_2 t_2 - r_4 + \frac{1}{8} p_2^2 + f(t_4 + \frac{1}{2} p_2 t_2)] g^4 \epsilon^{-2} + \dots, \end{aligned} \quad (\text{C.10})$$

$$\begin{aligned} c_{1(5)} &= f^{-1} V_{L(5)} + 1 - \alpha + f^{-1} \beta \\ &= f^{-1} V_{L(5)} + (-\frac{1}{2} p_2 r_2 + f^{-1} t_2) g^2 \epsilon^{-1} \\ &\quad + [-\frac{1}{2} p_4 - \frac{1}{2} p_2 t_2 - r_4 + \frac{1}{8} p_2^2 + f^{-1}(t_4 + \frac{1}{2} p_2 t_2)] g^4 \epsilon^{-2} + \dots, \end{aligned} \quad (\text{C.11})$$

where $V_{L(1)}$ and $V_{L(5)}$ are, for the $\psi\psi A$ vertex in Case A2, $V_{1L(1)}^{(2)}$ and $V_{1L(5)}^{(2)}$, of the form given by (A.19), (A.20). These values of $c_1 = c_{1(j=1)}^{(2)}$, $c_{1(5)} = c_{1(j=1)(5)}^{(2)}$, series in $g^{2n} \epsilon^{-n}$, renormalise the $\psi\psi A$ vertex matrix to the form (91) in Case A2.

We carry through a similar treatment of the $\psi\psi A$ vertex in Case B, with the first equation in (C.3) replaced by

$$(\Omega_{3L}^{-1})^{(B)} = 1 + \sum p_n^{(B)} w^{n+1} \epsilon^{-n} \quad (\text{C.12})$$

and with similar expansions in w for $(\Omega_{2L}^{-1})^{(B)}$, $\alpha^{(B)}$ and $\beta^{(B)}$ (the powers of g are still present, in the coefficients $p_n^{(B)}$, etc.). We obtain expressions for

$c_{1(j=1)}^{(B)}$, $c_{1(j=1)(5)}^{(B)}$ analogous to those for $c_{1(j=1)}^{(2)}$, $c_{1(j=1)(5)}^{(2)}$ given by (C.10), (C.11). A similar treatment goes through for Case A1, involving powers g^{2n} , g^{2n+1} .

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